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SPIN 1/2 PARTICLE IN THE FIELD OF THE DIRAC STRING ON THE BACKGROUND OF ANTI DE SITTER SPACE–TIME

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Abstract

The Dirac monopole string is specified for anti de Sitter cosmological model. Dirac equation for spin 1/2 particle in presence of this monopole has been examined on the background of anti de Sitter space-time in static coordinates. Instead of spinor monopole harmonics, the technique of Wigner D-functions is used. After separation of the variables radial equations have been solved exactly in terms of hypergeometric functions. The complete set of spinor wave solutions $\Psi_{\epsilon,j,m,\lambda}(t,r,\theta,\phi)$ has been constructed, the most attention is given to treating the states of minimal values for total moment quantum number j_{min}. At all values of j, the energy spectrum is discrete.
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1 Introduction

De Sitter and anti de Sitter geometrical models are given steady attention in the context of developing quantum theory in a curved space-time – for instance, see in [1]. In particular, the problem of description of the particles with different spins on these curved backgrounds has a long history – see [2–34]. Here we will be interested mostly in treating the Dirac equation in de Sitter model. In the present paper, the influence of the Dirac monopole string on the spin 1/2 particle in the anti de Sitter cosmological model is investigated. Instead of spinor monopole harmonics, the technique of Wigner D-functions is used. After separation of the variables radial equation have been solved exactly in terms of hypergeometric functions. The complete set of spinor wave solutions $\Psi_{\epsilon,j,m,\lambda}(t,r,\theta,\phi)$ has been constructed. Special attention is given to treating the states of minimal values for total moment quantum number j_{min}, these states turn to be much more complicated than in the flat Minkowski space. At all values of j, the energy spectrum is discrete.

2 Dirac particle in the anti de Sitter space

The Dirac equation (the notation according to [39] is used)

$$\left[i\gamma^{c}\,(e^{\alpha}_{(c)}\partial_{\alpha} + \frac{1}{2}\sigma^{ab}\gamma_{abc}) - M \right]\Psi = 0 \qquad (1)$$

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in static coordinates and tetrad of the anti de Sitter space-time

$$dS^2 = \Phi \; dt^2 - \frac{dr^2}{\Phi} - r^2(d\theta^2 + \sin^2\theta d\phi^2) \; , \quad \Phi = 1 + r^2 \; ,$$
$$e^{\alpha}_{(0)} = (\frac{1}{\sqrt{\Phi}}, 0, 0, 0) \; , \qquad e^{\alpha}_{(3)} = (0, \sqrt{\Phi}, 0, 0) \; ,$$
$$e^{\alpha}_{(1)} = (0, 0, \frac{1}{r}, 0) \; , \qquad e^{\alpha}_{(2)} = (1, 0, 0, \frac{1}{r\sin\theta}) \; ,$$
$$\gamma_{030} = \frac{\Phi'}{2\sqrt{\Phi}} \; , \; \gamma_{311} = \frac{\sqrt{\Phi}}{r} \; , \; \gamma_{322} = \frac{\sqrt{\Phi}}{r} \; , \; \gamma_{122} = \frac{\cos\theta}{r\sin\theta} \; , \tag{2}$$

takes the form

$$\Big[i\frac{\gamma^0}{\sqrt{\Phi}}\partial_t + i\sqrt{\Phi}\Big(\gamma^3\partial_r + \frac{\gamma^1\sigma^{31} + \gamma^2\sigma^{32}}{r} + \frac{\Phi'}{2\Phi}\gamma^0\sigma^{03}\Big) + \frac{1}{r}\Sigma_{\theta,\phi} - M \Big] \Psi(x) = 0 \; , \tag{3}$$

where

$$\Sigma_{\theta,\phi} = \; i\,\gamma^1\partial_\theta + \gamma^2\frac{i\partial + i\sigma^{12}\cos\theta}{\sin\theta} \; .$$

Eq. (3) reads

$$\Big[i\,\frac{\gamma^0}{\sqrt{\Phi}}\partial_t \; + i\sqrt{\Phi}\gamma^3 \; (\; \partial_r + \frac{1}{r} + \frac{\Phi'}{4\Phi} \;) + \frac{1}{r}\;\Sigma_{\theta,\phi} - M \Big] \Psi(x) = 0 \; . \tag{4}$$

From (4), with the substitution $\Psi(x) = r^{-1}\Phi^{-1/4}\;\psi(x)$, we get

$$\Big(i\,\frac{\gamma^0}{\sqrt{\Phi}}\partial_t \; + i\sqrt{\Phi}\gamma^3 \;\; \partial_r + \frac{1}{r}\;\Sigma_{\theta,\phi} - M \; \Big) \psi(x) = 0 \; . \tag{5}$$

Below the spinor basis will be used

$$\gamma^0 = \left| \begin{array}{cc} 0 & I \\ I & 0 \end{array} \right| , \; \gamma^j = \left| \begin{array}{cc} 0 & -\sigma_j \\ \sigma_j & 0 \end{array} \right| , \; i\sigma^{12} = \left| \begin{array}{cc} \sigma_3 & 0 \\ 0 & \sigma_3 \end{array} \right| .$$

3 Separation of the variables

Let us start with the monopole Abelian potential in the Schwinger's form [35] in flat Minkowski space

$$A^a(x) = (A^0, \; A^i) = \left(0 \; , \; g\;\frac{(\vec{r}\times\vec{n})\;(\vec{r}\,\vec{n})}{r\;(r^2 - (\vec{r}\,\vec{n})^2)}\right) . \tag{6}$$

Specifying $\vec{n} = (0,0,1)$ and translating the A_α to the spherical coordinates, we get

$$A_0 = 0 \; , \quad A_r = 0 \; , \quad A_\theta = 0 \; , \qquad A_\phi = g\;\cos\theta \; . \tag{7}$$

It is easily verified that this potentialA_ϕ obeys Maxwell equations in anti de Sitter space

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\alpha}\sqrt{-g}F^{\alpha\beta}=0\;,\;\sqrt{-g}=r^2\sin\theta\;,$$
$$F_{\phi\theta}=g\sin\theta\;,\qquad \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}r^2\sin\theta\;\frac{1}{r^2}\frac{1}{r^2\sin^2\theta}\;g\sin\theta=0\;. \tag{8}$$

Correspondingly, the Dirac equation in this electromagnetic field takes the form

$$\left[i\;\frac{\gamma^0}{\sqrt{\Phi}}\partial_t\;+i\sqrt{\Phi}\gamma^3\;\;\partial_r+\frac{1}{r}\;\Sigma^k_{\theta,\phi}-M\right]\psi(x)=0\;, \tag{9}$$

where

$$\Sigma^k_{\theta,\phi}=i\gamma^1\partial_\theta+\gamma^2\frac{i\partial_\phi+(i\sigma^{12}-k)\cos\theta}{\sin\theta}\;, \tag{10}$$

and $k\equiv eg/hc$. As readily verified, the wave operator in (9) commutes with the following three ones

$$J^k_1=l_1+\frac{(i\sigma^{12}-k)\cos\phi}{\sin\theta}\;,$$
$$J^k_2=\;l_2+\frac{(i\sigma^{12}-k)\sin\phi}{\sin\theta}\;,\qquad J^k_3=l_3\;, \tag{11}$$

which obey the $su(2)$ Lie algebra. Clearly, this monopole situation come entirely under the Schrödinger [36] and Pauli [37] approach (detailed treatment of the method was given in [40]). Correspondingly to diagonalizing the $\vec{J}^2_k$ and J^k_3, the function ψ is to be taken as ($D_\sigma\equiv D^j_{-m,\sigma}(\phi,\theta,0)$ stands for Wigner functions [38])

$$\psi^k_{\epsilon jm}(t,r,\theta,\phi)=e^{-i\epsilon t}\left|\begin{array}{l} f_1\;D_{k-1/2}\\ f_2\;D_{k+1/2}\\ f_3\;D_{k-1/2}\\ f_4\;D_{k+1/2}\end{array}\right|\;. \tag{12}$$

Further, with the he;p of recursive relations [38]

$$\partial_\theta D_{k+1/2}=aD_{k-1/2}-bD_{k+3/2}\;,\;\partial_\theta D_{k-1/2}=cD_{k-3/2}-aD_{k+1/2}\;,$$
$$\sin^{-1}\theta\,[\,-m-(k+1/2)\cos\theta\,]\;D_{k+1/2}=(-aD_{k-1/2}-bD_{k+3/2})\;\;,$$
$$\sin^{-1}\theta\,[\,-m-(k-1/2)\cos\theta\,]\;D_{k-1/2}=(-cD_{k-3/2}-aD_{k+1/2})\;,$$

$$b=\frac{\sqrt{(j-k-1/2)(j+k+3/2)}}{2}\;,$$
$$c=\frac{\sqrt{(j+k-1/2)(j-k+3/2)}}{2}\;,$$
$$a=\frac{1}{2}\sqrt{(j+1/2)^2-k^2}$$

we find how the $\Sigma^k_{\theta,\phi}$ acts on $\psi^k_{\epsilon jm}$

$$\Sigma^k_{\theta,\phi}\, \psi^k_{\epsilon jm} = i\sqrt{(j+1/2)^2 - k^2}\;\; e^{-i\epsilon t} \left| \begin{array}{c} -f_4\; D_{k-1/2} \\ +f_3\; D_{k+1/2} \\ +f_2\; D_{k-1/2} \\ -f_1\; D_{k+1/2} \end{array} \right| ; \tag{13}$$

hereafter the factor $\sqrt{(j+1/2)^2 - k^2}$ will be denoted by ν. For the $f_i(r)$, the radial system derived is

$$\begin{aligned}
&\frac{\epsilon}{\sqrt{\Phi}}\, f_3 \;-\; i\,\sqrt{\Phi}\frac{d}{dr}\, f_3 \;-\; i\,\frac{\nu}{r}\, f_4 \;-\; M\, f_1 = 0\,, \\
&\frac{\epsilon}{\sqrt{\Phi}}\, f_4 \;+\; i\,\sqrt{\Phi}\frac{d}{dr}\, f_4 \;+\; i\,\frac{\nu}{r}\, f_3 \;-\; M\, f_2 = 0\,, \\
&\frac{\epsilon}{\sqrt{\Phi}}\, f_1 \;+\; i\,\sqrt{\Phi}\frac{d}{dr}\, f_1 \;+\; i\,\frac{\nu}{r}\, f_2 \;-\; M\, f_3 = 0\,, \\
&\frac{\epsilon}{\sqrt{\Phi}}\, f_2 \;-\; i\,\sqrt{\Phi}\frac{d}{dr}\, f_2 \;-\; i\,\frac{\nu}{r}\, f_1 \;-\; M\, f_4 = 0\,.
\end{aligned} \tag{14}$$

Else one operator can be diagonalized together with $i\partial_t, \vec{J}^2_k, J^k_3$: namely, a generalized Dirac operator

$$\hat{K}^k \;= -\,i\,\gamma^0\,\gamma^3\,\Sigma^k_{\theta,\phi}\,. \tag{15}$$

From the equation $\hat{K}^k \psi_{\epsilon jm} = \lambda\; \psi_{\epsilon jm}$ we find two possible eigenvalues and restrictions on $f_i(r)$

$$f_4 = \delta\; f_1\,, \quad f_3 = \delta\; f_2\,, \qquad \lambda = -\delta\; \sqrt{(j+1/2)^2 - k^2}\,. \tag{16}$$

Correspondingly, the system (14) reduces to

$$\begin{aligned}
&\left(\sqrt{\Phi}\frac{d}{dr} + \frac{\nu}{r}\right) f \;+\; \left(\frac{\epsilon}{\sqrt{\Phi}} + \delta\; M\right)\; g = 0\,, \\
&\left(\sqrt{\Phi}\frac{d}{dr} - \frac{\nu}{r}\right) g \;-\; \left(\frac{\epsilon}{\sqrt{\Phi}} - \delta\; M\right)\; f = 0\,,
\end{aligned} \tag{17}$$

where

$$f = \frac{f_1 + f_2}{\sqrt{2}}\,, \qquad g = \frac{f_1 - f_2}{\sqrt{2}i}\,.$$

It is known that quantization of $k = eg/hc$ and j is given by

$$\begin{aligned}
&eg/hc = \pm 1/2,\; \pm 1,\; \pm 3/2, \ldots; \\
&j = \mid k \mid -1/2, \mid k \mid +1/2, \mid k \mid +3/2, \ldots
\end{aligned} \tag{18}$$

The case of minimal value $j_{min} = \mid k \mid -1/2$ must be treated separately in a special way. For example, let $k = +1/2$, then to the minimal value $j = 0$ there corresponds the wave function in terms of only (t, r)-dependent quantities

$$\psi^{(j=0)}_{k=+1/2}(x) = e^{-i\epsilon t} \left| \begin{array}{c} f_1(r) \\ 0 \\ f_3(r) \\ 0 \end{array} \right| . \tag{19}$$

At $k = -1/2$, we have

$$\psi^{(j=0)}_{k=-1/2}(x) = e^{-i\epsilon t} \left| \begin{array}{c} 0 \\ f_2(r) \\ 0 \\ f_4(r) \end{array} \right| . \qquad (20)$$

Thus, if $k = \pm 1/2$, then to the minimal values $j_{\min}$ there correspond the function substitutions which do not depend at all on the angular variables (θ, ϕ); at this point there exists some formal analogy between these electron-monopole states and S-states (with $l = 0$) for a boson field of spin zero: $\Phi_{l=0} = \Phi(r,t)$. However, it would be unwise to attach too much significance to this formal similarity because that (θ,ϕ)-independence of $(e-g)$-states is not a fact invariant under tetrad gauge transformations.

In contrast, the relation below (let $k = +1/2$)

$$\Sigma^{+1/2}_{\theta,\phi} \; \psi^{(j=0)}_{k=+1/2}(x) = \gamma^2 \cot\theta \; (i\sigma^{12} - 1/2) \; \psi^{(j=0)}_{k=+1/2} \equiv 0 \qquad (21)$$

is invariant under arbitrary tetrad gauge transformations. Correspondingly, the matter equation (9) takes on the form

$$\left(i \, \frac{\gamma^0}{\sqrt{\Phi}} \, \frac{\partial}{\partial t} + i \, \gamma^3 \sqrt{\Phi} \, \frac{\partial}{\partial r} - M \right) \psi^{(j=0)} = 0 \; . \qquad (22)$$

It is readily verified that both (19) and (20) representations are directly extended to $(e-g)$-states with $j = j_{\min}$ at all the other $k = \pm 1, \pm 3/2, \ldots$ Indeed,

$$k = +1, +3/2, +2, \ldots , \qquad \psi^{k>0}_{j_{min.}}(x) = e^{-i\epsilon t} \left| \begin{array}{l} f_1(r) \; D_{k-1/2} \\ 0 \\ f_3(r) \; D_{k-1/2} \\ 0 \end{array} \right| ; \qquad (23)$$

$$k = -1, -3/2, -2, \ldots , \qquad \psi^{k<0}_{j_{min.}}(x) = e^{-i\epsilon t} \left| \begin{array}{l} 0 \\ f_2(r) \; D_{k+1/2} \\ 0 \\ f_4(r) \; D_{k+1/2} \end{array} \right| , \qquad (24)$$

and the relation $\Sigma_{\theta,\phi} \psi_{j_{\min}} = 0$ still holds. For instance, let us consider in more detail the case of positive k. Using the recursive relations

$$\partial_\theta D_{k-1/2} = \frac{1}{2} \sqrt{2k-1} \; D_{k-3/2} \; ,$$

$$\sin^{-1}\theta \, [\, -m - (k-1/2)\cos\theta \,] \; D_{k-1/2} = -\frac{1}{2} \sqrt{2k-1} \; D_{k-3/2} \; ,$$

we get

$$i\gamma^1 \; \partial_\theta \left| \begin{array}{c} f_1(r) \; D_{k-1/2} \\ 0 \\ f_3(r) \; D_{k-1/2} \\ 0 \end{array} \right| = \frac{i}{2} \sqrt{2k-1} \left| \begin{array}{c} 0 \\ -f_3(r) \; D_{k-3/2} \\ 0 \\ +f_1(r) \; D_{k-3/2} \end{array} \right| ,$$

$$\gamma^2 \; \frac{i\partial_\phi + (i\sigma^{12} - k)\cos\theta}{\sin\theta} \left| \begin{array}{c} f_1(r) \; D_{k-1/2} \\ 0 \\ f_3(r) \; D_{k-1/2} \\ 0 \end{array} \right| = \frac{i}{2} \sqrt{2k-1} \left| \begin{array}{c} 0 \\ +f_3(r) \; D_{k-3/2} \\ 0 \\ -f_1(r) \; D_{k-3/2} \end{array} \right| ;$$

in a sequence, the identity $\Sigma_{\theta,\phi}\ \psi_{j_{\min}} \equiv 0$ holds. The case of negative k can be considered in the same way. Thus, at every k, the $j_{\min}$-state equation has the same unique form

$$\left(i\,\frac{\gamma^0}{\sqrt{\Phi}}\,\frac{\partial}{\partial t} \;+\; i\gamma^3 \sqrt{\Phi}\,\frac{\partial}{\partial r} \;-\; M \right) \psi_{j_{mi}} = 0\ ; \tag{25}$$

which leads to the same unique radial system

$k = +1/2, +1, \ldots$

$$\begin{aligned} &\frac{\epsilon}{\sqrt{\Phi}}\ f_3 - i\ \sqrt{\Phi}\frac{d}{dr}\ f_3 - M\ f_1 = 0\ , \\ &\frac{\epsilon}{\sqrt{\Phi}}\ f_1 + i\ \sqrt{\Phi}\frac{d}{dr}\ f_1 - M\ f_3 = 0\ ; \end{aligned} \tag{26}$$

$k = -1/2, -1, \ldots$

$$\begin{aligned} &\frac{\epsilon}{\sqrt{\Phi}}\ f_4 + i\ \sqrt{\Phi}\frac{d}{dr}\ f_4 - M\ f_2 = 0\ , \\ &\frac{\epsilon}{\sqrt{\Phi}}\ f_2 - i\ \sqrt{\Phi}\frac{d}{dr}\ f_2 - M\ f_4 = 0\ . \end{aligned} \tag{27}$$

In the limit of flat space–time, these equations are equivalent respectively to
$k = +1/2, +1, \ldots$

$$\left(\frac{d^2}{dr^2} + \epsilon^2 - m^2 \right) f_1 = 0\ , \quad f_3 = \frac{1}{m}\left(\epsilon + i\frac{d}{dr} \right) f_1\ ; \tag{28}$$

$k = -1/2, -1, \ldots$

$$\left(\frac{d^2}{dr^2} + \epsilon^2 - m^2 \right) f_4 = 0\ , \quad f_2 = \frac{1}{m}\left(\epsilon + i\frac{d}{dr} \right)\ f_4\ . \tag{29}$$

These equation both lead us to the functions $f = \exp(\pm\sqrt{m^2 - \epsilon^2}\ r)$. In particular, at $\epsilon\ <\ m$, we have a solution

$$\exp\ (\ -\sqrt{m^2 - \epsilon^2}\ r\ )\ , \tag{30}$$

which seems to be appropriate to describe bound states in the electron-monopole system.

4 Solution of the radial equations

Let us turn back to the system (17) and (for definiteness) consider equations at $\delta = +1$ (formally the second case $\delta = -1$ corresponds to the change $M \Longrightarrow -M$)

$$\begin{aligned} &(\sqrt{\Phi}\frac{d}{dr}\ +\ \frac{\nu}{r}\ )\ f\ +\ (\frac{\epsilon}{\sqrt{\Phi}}\ +\ M)\ g\ =\ 0\ , \\ &(\sqrt{\Phi}\frac{d}{dr}\ -\ \frac{\nu}{r}\ )\ g\ -\ (\frac{\epsilon}{\sqrt{\Phi}}\ -\ M)\ f\ =\ 0\ . \end{aligned} \tag{31}$$

Here we see additional singularities at the points

$$\epsilon + \sqrt{\Phi}\, M = 0 \qquad \text{or} \qquad \epsilon - \sqrt{\Phi}\, M = 0 \, .$$

For instance, the equation for $f(r)$ has the form

$$\frac{d^2}{dr^2} f + \left(\frac{2r}{1+r^2} - \frac{Mr}{\sqrt{1+r^2}(\epsilon + M\sqrt{1+r^2})} \right) \frac{d}{dr} f + \left(\frac{\epsilon^2}{(1+r^2)^2} - \frac{M^2}{1+r^2} \right.$$
$$\left. - \frac{\nu^2}{r^2(1+r^2)} - \frac{\nu}{r^2(1+r^2)^{3/2}} - \frac{M\nu}{(1+r^2)(\epsilon + M\sqrt{1+r^2})} \right) f = 0 \, .$$

However, there exists possibility to move these singularities away through a special transformation of the functions $f(r), g(r)$ [24]. To this end, let us introduce a new variable $r = \sinh \rho$, eqs. (31) look simpler

$$\begin{aligned} &(\frac{d}{d\rho} + \frac{\nu}{\sinh \rho}) f + (\frac{\epsilon}{\cosh \rho} + M) g = 0 \, , \\ &(\frac{d}{d\rho} - \frac{\nu}{\sinh \rho}) g - (\frac{\epsilon}{\cosh \rho} - M) f = 0 \, . \end{aligned} \tag{32}$$

Summing and subtracting two last equations, we get

$$\begin{aligned} &\frac{d}{d\rho}(f+g) + \frac{\nu}{\sinh \rho}(f-g) - \frac{\epsilon}{\cosh \rho}(f-g) + M(f+g) = 0 \, , \\ &\frac{d}{d\rho}(f-g) + \frac{\nu}{\sinh \rho}(f+g) + \frac{\epsilon}{\cosh \rho}(f+g) - M(f-g) = 0 \, . \end{aligned} \tag{33}$$

Introducing two new functions

$$f + g = e^{-\rho/2}(F+G) \, , \qquad f - g = e^{+\rho/2}(F-G) \, , \tag{34}$$

or in matrix form

$$\left| \begin{array}{c} G \\ H \end{array} \right| = \left| \begin{array}{cc} \cosh \rho/2 & -\sinh \rho/2 \\ -\sinh \rho/2 & \cosh \rho/2 \end{array} \right| \left| \begin{array}{c} g \\ h \end{array} \right| , \tag{35}$$

where (see definition of the variable z below)

$$\cosh \frac{\rho}{2} = \sqrt{\frac{\sqrt{1-z}+1}{2}} \, , \qquad \sinh \frac{\rho}{2} = \sqrt{\frac{\sqrt{1-z}-1}{2}} \, , \tag{36}$$

one transforms (33) into

$$\begin{aligned} &\frac{d}{d\rho} e^{-\rho/2}(F+G) + \frac{\nu}{\sinh \rho} e^{+\rho/2}(F-G) \\ &- \frac{\epsilon}{\cosh \rho} e^{+\rho/2}(F-G) + M e^{-\rho/2}(F+G) = 0 \, , \\ &\frac{d}{d\rho} e^{+\rho/2}(F-G) + \frac{\nu}{\sinh \rho} e^{-\rho/2}(F+G) \\ &+ \frac{\epsilon}{\cos \rho} e^{-\rho/2}(F+G) - M e^{+\rho/2}(F-G) = 0 \, , \end{aligned}$$

or

$$\frac{d}{d\rho}(F+G) - \frac{1}{2}(F+G) + \frac{\nu}{\sinh\rho}(\cosh\rho+\sinh\rho)(F-G)$$
$$-\frac{\epsilon}{\cosh\rho}(\cosh\rho+\sinh\rho)(F-G) + M(F+G) = 0\,,$$

$$\frac{d}{d\rho}(F-G) + \frac{1}{2}(F-G) + \frac{\nu}{\sinh\rho}(\cosh\rho-\sinh\rho)(F+G)$$
$$+\frac{\epsilon}{\cosh\rho}(\cosh\rho-\sinh\rho)(F+G) - M(F-G) = 0\,.$$

Now summing and subtracting two last equations, we obtain

$$(\frac{d}{d\rho} + \nu\,\frac{\cosh\rho}{\sinh\rho} - \epsilon\,\frac{\sinh\rho}{\cosh\rho})\;F + \;(\,\epsilon + M - \nu - \frac{1}{2})\;G = 0\,,$$
$$(\frac{d}{d\rho} - \nu\,\frac{\cosh\rho}{\sinh\rho} + \epsilon\,\frac{\sinh\rho}{\cosh\rho})\;G + (-\epsilon + M + \nu - \frac{1}{2})\;F = 0\,. \tag{37}$$

Let us translate eqs. (37) to the variable z:

$$r^2 = \sinh^2\rho = -z, \qquad \frac{d}{d\rho} = 2\sqrt{-z(1-z)}\frac{d}{dz}\,,$$

$$\left(2\sqrt{-z(1-z)}\frac{d}{dz} + \nu\frac{\sqrt{1-z}}{\sqrt{-z}} - \epsilon\,\frac{\sqrt{-z}}{\sqrt{1-z}}\right)F$$
$$+(\epsilon + M - \nu - \frac{1}{2})G = 0\,,$$
$$\left(2\sqrt{-z(1-z)}\frac{d}{dz} - \nu\frac{\sqrt{1-z}}{\sqrt{-z}} + \epsilon\,\frac{\sqrt{-z}}{\sqrt{1-z}}\right)G$$
$$+(-\epsilon + M + \nu - \frac{1}{2})F = 0\,. \tag{38}$$

From (38) it follow two 2-nd order differential equations for F and G respectively

$$z(1-z)\frac{d^2F}{dz^2} + (\frac{1}{2} - z)\frac{dF}{dz}$$
$$+\left[\frac{1}{4}\left(M - \frac{1}{2}\right)^2 - \frac{\epsilon(\epsilon-1)}{4(1-z)} - \frac{\nu(\nu+1)}{4z}\right]F = 0,$$
$$z(1-z)\frac{d^2G}{dz^2} + (\frac{1}{2} - z)\frac{dG}{dz}$$
$$+\left[\frac{1}{4}\left(M - \frac{1}{2}\right)^2 - \frac{\epsilon(\epsilon+1)}{4(1-z)} - \frac{\nu(\nu-1)}{4z}\right]G = 0. \tag{39}$$

With the use of substitutions

$$F = z^A(1-z)^B\bar{F}(z)\,, \qquad G = z^K(1-z)^L\bar{G}(z)\,,$$

eqs. (39) take the form

$$z(1-z)\,\frac{d^2\bar{F}}{dz^2}+\left[2A+\frac{1}{2}-(2A+2B+1)z\right]\,\frac{d\bar{F}}{dz}$$
$$+\left[\frac{1}{4}\left(M-\frac{1}{2}\right)^2-(A+B)^2-\frac{\epsilon(\epsilon-1)+2B(1-2B)}{4(1-z)}\right.$$
$$\left.-\frac{\nu(\nu+1)-2A(2A-1)}{4z}\right]\bar{F}=0\,, \tag{40}$$

$$z(1-z)\,\frac{d^2\bar{G}}{dz^2}+\left[2K+\frac{1}{2}-(2K+2L+1)z\right]\,\frac{d\bar{G}}{dz}$$
$$+\left[\frac{1}{4}\left(M-\frac{1}{2}\right)^2-(K+L)^2-\frac{\epsilon(\epsilon+1)+2L(1-2L)}{4(1-z)}\right.$$
$$\left.-\frac{\nu(\nu-1)-2K(2K-1)}{4z}\right]\bar{G}=0\,. \tag{41}$$

First let us consider eq. (40); at A and B taken accordingly

$$A=\frac{1+\nu}{2}\,,\ -\frac{\nu}{2}\,,\qquad B=\frac{\epsilon}{2}\,,\ \frac{1-\epsilon}{2} \tag{42}$$

it becomes simpler

$$z(1-z)\,\frac{d^2f}{dz^2}+\left[2A+\frac{1}{2}-(2A+2B+1)z\right]\,\frac{df}{dz}$$
$$+\left[\frac{1}{4}\left(M-\frac{1}{2}\right)^2-(A+B)^2\right]f=0\,, \tag{43}$$

which is of hypergeometric type with parameters

$$a=\frac{M}{2}-\frac{1}{4}+A+B\,,\qquad b=-\frac{M}{2}+\frac{1}{4}+A+B\,,\qquad c=2A+1/2\,.$$

To construct functions appropriate to describe bound states we must choose

$$A=\frac{1+\nu}{2}>0\,,\qquad B=\frac{1-\epsilon}{2}<0\,,\qquad c=\nu+3/2\,; \tag{44}$$

polynomial solutions will arise with the quantization rule imposed

$$a=-n\,,\qquad \epsilon_n=M+2n+\nu+\frac{3}{2}\,,$$
$$b=-n-M-1/2\,,\qquad c=\nu+3/2\,. \tag{45}$$

Now let us turn to eq. (41). At A, B chosen according to

$$K=\frac{1-\nu}{2}\,,\ \frac{\nu}{2}\,,\qquad L=-\frac{\epsilon}{2}\,,\ \frac{1+\epsilon}{2} \tag{46}$$

it will be simpler

$$z(1-z)\,\frac{d^2g}{dz^2}+\left[2K+\frac{1}{2}-(2K+2L+1)z\right]\,\frac{dg}{dz}$$
$$+\left[\frac{1}{4}\left(M-\frac{1}{2}\right)^2-(K+L)^2\right]g=0\,, \tag{47}$$

which is of hypergeometric type

$$\alpha=\frac{M}{2}-\frac{1}{4}+K+L\,,\qquad \beta=-\frac{M}{2}+\frac{1}{4}+K+L\,,\qquad \gamma=2K+\frac{1}{2}\,.$$

Again, to get bound states we choose the values

$$K=\frac{\nu}{2}>0\,,\qquad L=-\frac{\epsilon}{2}<0\,, \tag{48}$$

then the quantization rule arises

$$\alpha=-N\,,\qquad \epsilon_N=M+2N+\nu-\frac{1}{2}\,. \tag{49}$$

It can be noted that $\epsilon_N=\epsilon_n$, when $N=n+1$.

Let us calculate relative coefficient between functions $F(z)$ and $G(z)$. These being taken in the form

$$F(z)=F_0\,z^{(1+\nu)/2}\,(1-z)^{(1-\epsilon)/2}\bar{F}(a,b,c;z)\,,\qquad c=\frac{3}{2}+\nu\,,$$
$$a=\frac{M}{2}+\frac{3}{4}+\frac{\nu}{2}-\frac{\epsilon}{2}\,,\qquad b=-\frac{M}{2}+\frac{5}{4}+\frac{\nu}{2}-\frac{\epsilon}{2}\,; \tag{50}$$

and

$$G(z)=G_0\,z^{\nu/2}\,(1-z)^{-\epsilon/2}\bar{G}(\alpha,\beta,\gamma;z)\,,\quad \gamma=\frac{1}{2}+\nu=c-1\,,$$
$$\alpha=\frac{M}{2}-\frac{1}{4}+\frac{\nu}{2}-\frac{\epsilon}{2}=a-1\,,\qquad \beta=-\frac{M}{2}+\frac{1}{4}+\frac{\nu}{2}-\frac{\epsilon}{2}=b-1\,, \tag{51}$$

must obey the following system

$$\left(2\sqrt{-z(1-z)}\frac{d}{dz}+\nu\frac{\sqrt{1-z}}{\sqrt{-z}}-\epsilon\,\frac{\sqrt{-z}}{\sqrt{1-z}}\right)F+(+\epsilon+M-\nu-\frac{1}{2})G=0\,,$$
$$\left(2\sqrt{-z(1-z)}\frac{d}{dz}-\nu\,\frac{\sqrt{1-z}}{\sqrt{-z}}+\epsilon\frac{\sqrt{-z}}{\sqrt{1-z}}\right)G+(-\epsilon+M+\nu-\frac{1}{2})F=0\,.$$

To find a relative factor, it is convenient to use the second equation

$$\left(-2\sqrt{-z(1-z)}\frac{d}{dz}-\nu\frac{\sqrt{1-z}}{\sqrt{-z}}+\epsilon\frac{\sqrt{-z}}{\sqrt{1-z}}\right)G$$
$$+(-\epsilon+M+\nu-\frac{1}{2})F=0\,.$$

Substituting expressions for F and G, after simple calculation we get to

$$2\,iG_0\,\frac{d\bar{G}}{dz} = F_0\,(-\epsilon + M + \nu - \frac{1}{2})\,\bar{F}\ .$$

Allowing for the known rule for differentiating hypergeometric functions

$$\frac{d}{dz}\bar{G}(z) = \frac{d}{dz}F(a-1, b-1, c-1; z)$$
$$= \frac{(a-1)(b-1)}{c-1}F(a, b, c; z) = \frac{(a-1)(b-1)}{c-1}\bar{F}(z)\ ,$$

we obtain

$$2\,iG_0\,\frac{(a-1)(b-1)}{c-1} = F_0\,(-\epsilon + M + \nu - \frac{1}{2})\ .$$

Ultimately, we arrive at the formula

$$F_0 = i\,\frac{M - 1/2 + N}{2}\,G_0\ , \tag{52}$$

remembering that $\epsilon_N = M - 1/2 + 2N + \nu$.

5 Radial equations in the case j_{min}

Let us turn back to the case of the minimal value of j:

$k = +1/2, +1, \ldots$

$$\frac{\epsilon}{\sqrt{\Phi}}\ f_3 - i\,\sqrt{\Phi}\frac{d}{dr}\ f_3 - M\ f_1 = 0\ ,$$
$$\frac{\epsilon}{\sqrt{\Phi}}\ f_1 + i\,\sqrt{\Phi}\frac{d}{dr}\ f_1 - M\ f_3 = 0\ ; \tag{53}$$

from where for new functions

$$H\ =\ \frac{f_1 + f_3}{\sqrt{2}}\ , \qquad G\ =\ \frac{f_1 - f_3}{i\sqrt{2}}$$

we derive

$k = +1/2, +1, \ldots$

$$\sqrt{\Phi}\frac{d}{dr}H + (\frac{\epsilon}{\sqrt{\Phi}} + M)G = 0\ ,$$
$$\sqrt{\Phi}\frac{d}{dr}G - (\frac{\epsilon}{\sqrt{\Phi}} - M)H = 0\ . \tag{54}$$

And in the same manner for another case we have

$k = -1/2, -1, \ldots$

$$\frac{\epsilon}{\sqrt{\Phi}}\ f_4 + i\,\sqrt{\Phi}\frac{d}{dr}\ f_4 - M\ f_2 = 0\ ,$$
$$\frac{\epsilon}{\sqrt{\Phi}}\ f_2 - i\,\sqrt{\Phi}\frac{d}{dr}\ f_2 - M\ f_4 = 0\ , \tag{55}$$

from whence for new functions

$$H \;=\; \frac{f_2 + f_4}{\sqrt{2}} \; , \qquad G \;=\; \frac{f_2 - f_4}{i\sqrt{2}}$$

we obtain

$$\sqrt{\Phi}\frac{d}{dr}G + (\frac{\epsilon}{\sqrt{\Phi}} - M)H = 0 \; ,$$
$$\sqrt{\Phi}\frac{d}{dr}H - (\frac{\epsilon}{\sqrt{\Phi}} + M)G = 0 \; . \tag{56}$$

We can use the above method to eliminate nonphysical singular points. Let us perform special transformation on the functions

$$G + H = e^{-\rho/2}(g+h) \; , \qquad G - H = e^{+\rho/2}(g-h) \; . \tag{57}$$

After simple calculation we arrive at

<u>instead of (54)</u>

$$(\frac{d}{d\rho} + \epsilon\frac{\sinh\rho}{\cosh\rho})\; g + (\; -\epsilon + M - 1/2)\; h = 0 \; ,$$
$$(\frac{d}{d\rho} - \epsilon\frac{\sinh\rho}{\cosh\rho})\; h + (+\epsilon + M - 1/2)\; g = 0 \; ; \tag{58}$$

<u>instead of (56)</u>

$$(\frac{d}{d\rho} + \epsilon\;\frac{\sinh\rho}{\cosh\rho})\; h + (-\epsilon - M - 1/2)\; g = 0 \; ,$$
$$(\frac{d}{d\rho} - \epsilon\frac{\sinh\rho}{\cosh\rho})\; g + \;(+\epsilon - M - 1/2)\; h = 0 \; . \tag{59}$$

In the variable z

$$r = \sinh\rho = \sqrt{-z}$$

the system (58) takes the form

$$\sqrt{-z(1-z)}\left(\frac{d}{dz} - \frac{\epsilon/2}{1-z}\right) g - \frac{(-\epsilon + M - 1/2)}{2}\; h = 0 \; ,$$
$$\sqrt{-z(1-z)}\left(\frac{d}{dz} + \frac{\epsilon/2}{1-z}\right) h - \frac{(+\epsilon + M - 1/2)}{2}\; g = 0 \; . \tag{60}$$

Note that the system is symmetric with respect to changes

$$f \Longleftrightarrow h \; , \qquad \epsilon \Longleftrightarrow -\epsilon \; . \tag{61}$$

After excluding the function h from (60) we get

$$h = \frac{2}{(-\epsilon + M - 1/2)} \sqrt{(-z)(1-z)} \left(\frac{d}{dz} - \frac{\epsilon/2}{1-z} \right) g \; ,$$

$$\sqrt{(-z)(1-z)} \left(\frac{d}{dz} + \frac{\epsilon/2}{1-z} \right) \sqrt{(-z)(1-z)} \left(\frac{d}{dz} - \frac{\epsilon/2}{1-z} \right) g$$
$$- \frac{(M-1/2)^2 - \epsilon^2)}{4} \; g = 0 \; . \tag{62}$$

Ultimately, an equation for $g(z)$ reads

$$z(1-z) \frac{d^2 g}{dz^2} + (1/2 - z) \frac{dg}{dz} + \left(\frac{(M-1/2)^2}{4} - \frac{\epsilon^2 + \epsilon}{4} \; \frac{1}{1-z} \right) g = 0 \; . \tag{63}$$

In the same manner we get a second order differential equation for h after exclusion of g:

$$g = \frac{2}{(+\epsilon + M - 1/2)} \sqrt{(-z)(1-z)} \left(\frac{d}{dz} + \frac{\epsilon/2}{1-z} \right) h \; ,$$

$$\sqrt{(-z)(1-z)} \left(\frac{d}{dz} - \frac{\epsilon/2}{1-z} \right) \sqrt{(-z)(1-z)} \left(\frac{d}{dz} + \frac{\epsilon/2}{1-z} \right) h$$
$$- \frac{(M-1/2)^2 - \epsilon^2)}{4} \; h = 0 \; , \tag{64}$$

and ultimately

$$z(1-z) \frac{d^2 h}{dz^2} + (1/2 - z) \frac{dh}{dz} + \left(\frac{(M-1/2)^2}{4} - \frac{\epsilon^2 - \epsilon}{4} \; \frac{1}{1-z} \right) h = 0 \; . \tag{65}$$

Equations (65) and (63) differ only in the sign at the parameter ϵ.

6 Solutions of radial equations in the case j_{min}

With the use of substitution $g = (1-z)^A \varphi(z)$, from (63) we produce for φ

$$z(1-z) \; \varphi'' + [\frac{1}{2} - (1+2A)z] \; \varphi'$$
$$+ \left[\left(A^2 - \frac{A}{2} - \frac{\epsilon^2 + \epsilon}{4} \right) \frac{1}{1-z} - A^2 + \frac{(M-1/2)^2}{4} \right] \varphi \; . \tag{66}$$

Requiring

$$A^2 - \frac{A}{2} - \frac{\epsilon^2 + \epsilon}{4} = 0 \qquad \Longrightarrow \qquad 2A = \epsilon + 1, -\epsilon$$

one gets

$$z(1-z) \; \varphi'' + [\frac{1}{2} - (1+2A)z] \; \varphi' - \frac{4A^2 - (M-1/2)^2}{4} \; \varphi = 0 \; ,$$
$$\varphi = F(a,b,c,z) \; , \; c = \frac{1}{2} \; , \; a + b = 2A, \; ab = \frac{4A^2 - (M-1/2)^2}{4} \; , \tag{67}$$

that is

$$a = \frac{2A + (M - 1/2)}{2} , \qquad b = \frac{2A - (M - 1/2)}{2} . \tag{68}$$

Below we will use negative values for A

$$A = -\epsilon/2 , \qquad g(z) = (1-z)^{-\epsilon/2} \varphi(z) ; \tag{69}$$

so that

$$a = \frac{-\epsilon + (M - 1/2)}{2} , \qquad b = \frac{-\epsilon - (M - 1/2)}{2} . \tag{70}$$

Any 2-nd order differential equation has two linearly independent solutions; here they are

$$\varphi_1 = U_1(z) = F(a, b, c; z) ,$$

$$\varphi_2 = U_5(z) = z^{1-c} F(a + 1 - c, b + 1 - c, 2 - c; z) . \tag{71}$$

Similar analysis can be performed for eq. (65)

$$z(1-z) \frac{d^2 h}{dz^2} + (1/2 - z) \frac{dh}{dz} + \left(\frac{(M - 1/2)^2}{4} - \frac{\epsilon^2 - \epsilon}{4} \; \frac{1}{1-z} \right) h = 0 . \tag{72}$$

With the use of substitution $h(z) = (1-z)^L \eta(z)$, for $\eta(z)$ we produce

$$z(1-z) \; \eta'' + [\frac{1}{2} - (1 + 2L)z] \; \eta'$$
$$+ \left[\left(L^2 - \frac{L}{2} - \frac{\epsilon^2 - \epsilon}{4} \right) \frac{1}{1-z} - L^2 + \frac{(M - 1/2)^2}{4} \right] \eta . \tag{73}$$

Requiring

$$L^2 - \frac{L}{2} - \frac{\epsilon^2 - \epsilon}{4} = 0 \qquad \Longrightarrow \qquad 2L = +\epsilon, -\epsilon + 1$$

one gets

$$z(1-z) \; \eta'' + [\frac{1}{2} - (1 + 2L)z] \; \eta' - \frac{4L^2 - (M - 1/2)^2}{4} \; \eta = 0 ,$$
$$\eta = F(\alpha, \beta, \gamma, z) , \qquad \gamma = \frac{1}{2} ,$$
$$\alpha + \beta = 2L, \qquad \alpha\beta = \frac{4L^2 - (M - 1/2)^2}{4} , \tag{74}$$

that is

$$\alpha = \frac{2L - (M - 1/2)}{2} , \qquad \beta = \frac{2L + (M - 1/2)}{2} . \tag{75}$$

Below we will use negative values for L

$$L = (-\epsilon + 1)/2 < 0 \, , \qquad h(z) = (1-z)^{(-\epsilon+1)/2} \eta(z) \, , \tag{76}$$

so that

$$\alpha = \frac{-\epsilon + 1 + (M - 1/2)}{2} \, , \qquad \beta = \frac{-\epsilon + 1 - (M - 1/2)}{2} \, . \tag{77}$$

Functions $g(z)$ and $h(z)$ must obey the above system of first order differential equations. To verify that, let us start with the functions

$$\begin{gathered} g = G_0 (1-z)^A \varphi_1(z) \, , \qquad 2A = -\epsilon \, , \\ \varphi_1 = F(a,b,c,z) \qquad c = 1/2 \, , \\ a = \frac{-\epsilon + (M - 1/2)}{2} \, , \qquad b = \frac{-\epsilon - (M - 1/2)}{2} \, , \end{gathered} \tag{78}$$

$$\begin{gathered} h = H_0 (1-z)^L \eta_2(z) \, , \qquad 2L = -\epsilon + 1 \, , \\ \eta_2 = z^{1/2} F(\alpha + 1 - \gamma \, , \beta + 1 - \gamma, 2 - \gamma, z) \, , \\ \alpha + 1 - \gamma = \frac{-\epsilon + 2 + (M - 1/2)}{2} = a + 1 \, , \\ \beta + 1 - \gamma = \frac{-\epsilon + 2 - (M - 1/2)}{2} = b + 1 \, , \\ 2 - \gamma = c + 1 \, , \end{gathered} \tag{79}$$

and relate them with the help of the first equation in the system (60)

$$\sqrt{(-z)(1-z)} \left(\frac{d}{dz} - \frac{\epsilon/2}{1-z} \right) g - \frac{(-\epsilon + M - 1/2)}{2} \, h = 0 \, .$$

After simple calculations we obtain

$$G_0 i \frac{d}{dz} F(a,b,c,z) = H_0 \frac{(-\epsilon + M - 1/2)}{2} F(a+1, b+1, c+1, z) \, ,$$

from whence it follows

$$G_0 i \frac{ab}{c} = H_0 \frac{(-\epsilon + M - 1/2)}{2} \, ,$$

that is

$$H_0 = i \, (-\epsilon - M + 1/2) \, G_0 \, .$$

To get polynomial solutions we must require

$$\begin{gathered} a = -n \quad \Longrightarrow \quad \epsilon_n = M + 2n - 1/2 \, , \\ b = -n - M + 1/2 \, , \qquad c = 1/2 \, , \\ g(z) = (1-z)^{-(\epsilon_n + 1)/2} F(a,b,c,z) \, . \end{gathered} \tag{80}$$

note that

$$g(z) = (1-z)^{-n-(M-1/2)/2} \, {}'F(-n, -n - M + \frac{1}{2}, \frac{1}{2}, z) \, ; \tag{81}$$

therefore as $z = -r^2 \longrightarrow -\infty$ the function $g(z)$ tends to zero

$$g(z) \longrightarrow 0 \; , \qquad \text{only} \quad \text{if} \qquad M > \frac{1}{2} \; .$$

In usual units, that condition for existence of bound states consistent with anti de Sitter geometry structure, the inequality $M > \frac{1}{2}$, looks as

$$\rho > \frac{1}{2} \frac{\hbar}{Mc} = \frac{1}{2} \lambda_e = 1.213 \times 10^{-12} \; metre$$

so it can be broken only in a very strong anti de Sitter gravitation background, the latter is beyond of our treatment.

Let us write down several energy levels (in usual units)

$$\epsilon_0 = Mc^2 - \frac{1}{2} \frac{c\hbar}{\rho} \; , \qquad \epsilon_1 = Mc^2 + \frac{3}{2} \frac{c\hbar}{\rho} \; , \qquad \epsilon_2 = Mc^2 + \frac{5}{2} \frac{c\hbar}{\rho} \; , \; ... \tag{82}$$

or

$$\epsilon_0 = Mc^2 (1 - \frac{1}{2} \frac{\lambda_e}{\rho}) \; , \qquad \epsilon_1 = Mc^2 (1 + \frac{3}{2} \frac{\lambda_e}{\rho}) \; , \qquad \epsilon_2 = Mc^2 (1 + \frac{5}{2} \frac{\lambda_e}{\rho}) \; , \; ... \tag{83}$$

If one mentally increases the curvature radius ρ, the energy levels will become denser and the minimal level tends to the value Mc^2

$$\epsilon_0 = Mc^2 (1 - \frac{1}{2} \frac{\lambda_e}{\rho}) \quad \longrightarrow \quad Mc^2 \; . \tag{84}$$

7 Conclusions and discussion

To understand better results, let us discuss the case of minimal j_{min} in the limit of vanishing curvature. To this end, let us specify in more detail solutions for minimal values j_{min} in Minkowski space:

$k = +1/2, +1, \ldots$

$$\begin{aligned} &\epsilon \; f_3 - i \; \frac{d}{dr} \; f_3 - M \; f_1 = 0 \; , \\ &\epsilon \; f_1 + i \; \frac{d}{dr} \; f_1 - M \; f_3 = 0 \; ; \end{aligned} \tag{85}$$

$k = -1/2, -1, \ldots$

$$\begin{aligned} &\epsilon \; f_4 + i \; \frac{d}{dr} \; f_4 - M \; f_2 = 0 \; , \\ &\epsilon \; f_2 - i \; \frac{d}{dr} \; f_2 - M \; f_4 = 0 \; . \end{aligned} \tag{86}$$

Let detail the case of positive $k = +1/2, +1, \ldots$. Let it be

$$\frac{f_1 + f_3}{\sqrt{2}} = h(r) \; , \qquad \frac{f_1 - f_3}{i\sqrt{2}} = g(r) \tag{87}$$

relevant equations are

$$\frac{d}{dr}h + (\epsilon + M)\; g = 0 \; , \qquad \frac{d}{dr}g - (\epsilon - M)h = 0 \; . \tag{88}$$

With the substitutions

$$h(r) = He^{\gamma r} \; , \qquad g(r) = Ge^{\gamma r} \tag{89}$$

we get (first let it be $(\epsilon^2 - M^2) > 0$)

$$\begin{gathered} \gamma^2 = -(\epsilon^2 - M^2) \equiv = -p^2 \; , \qquad \gamma = +ip, -ip \; . \\ H\gamma + (\epsilon + M)G = 0 \qquad \text{or} \qquad G\gamma - (\epsilon - M)H = 0 \; . \end{gathered} \tag{90}$$

Thus we have two linearly independent solutions

$$\begin{aligned} h_1(r) = H_1 e^{+ipr} \; , \qquad g_1(r) = G_1 e^{+ipr} \; , \qquad G_1 = \frac{\epsilon - M}{ip} H_1 \; ; \\ h_2(r) = H_2 e^{-ipr} \; , \qquad g_2(r) = G_2 e^{-ipr} \; , \qquad G_2 = \frac{\epsilon - M}{-ip} H_2 \; . \end{aligned} \tag{91}$$

Below, we take $H_1 = H_2 = 1$. We can introduce two linear combinations of these solutions the first

$$\begin{gathered} \frac{h_1(r) + h_2(r)}{2} = \cos pr \; , \\ \frac{g_1(r) + g_2(r)}{2} = \frac{\epsilon - M}{p} \; \sin pr \; ; \end{gathered} \tag{92}$$

the second

$$\begin{gathered} \frac{h_1(r) - h_2(r)}{2i} = \sin pr \; , \\ \frac{g_1(r) - g_2(r)}{2i} = \frac{\epsilon - M}{-p} \; \cos pr \; . \end{gathered} \tag{93}$$

Now let us specify the case $(\epsilon^2 - M^2) < 0)$

$$\begin{gathered} \gamma^2 = -(\epsilon^2 - M^2) \equiv = +q^2 \; , \qquad \gamma = +q, -q \; . \\ H\gamma + (\epsilon + M)G = 0 \qquad \text{or} \qquad G\gamma - (\epsilon - M)H = 0 \; . \end{gathered} \tag{94}$$

Thus we have two linearly independent solutions

$$\begin{aligned} h_1(r) = H_1 e^{+qr} \; , \qquad g_1(r) = G_1 e^{+qr} \; , \qquad G_1 = \frac{\epsilon - M}{q} H_1 \; ; \\ h_2(r) = H_2 e^{-qr} \; , \qquad g_2(r) = G_2 e^{-qr} \; , \qquad G_2 = \frac{\epsilon - M}{-q} H_2 \; . \end{aligned} \tag{95}$$

Below, we take $H_1 = H_2 = 1$. We can introduce two linear combinations of these solutions the first

$$\frac{h_1(r) + h_2(r)}{2} = \cosh qr \ ,$$
$$\frac{g_1(r) + g_2(r)}{2} = \frac{\epsilon - M}{q} \sinh qr \tag{96}$$

the second

$$\frac{h_1(r) - h_2(r)}{2} = \sinh qr \ ,$$
$$\frac{g_1(r) - g_2(r)}{2} = \frac{\epsilon - M}{q} \cosh qr \ . \tag{97}$$

Evidently, above constructed solutions in de Sitter model provide us with generalizations of these of Minkowski space. It may be verified additionally by direct limiting process when $\rho \to \infty$. To this end, let us translate solutions in de Sitter space to usual units

$$g_1(R) = \left(1 + \frac{R^2}{\rho^2}\right)^{-\frac{E\rho}{2c\hbar}} F(a, b, c; -\frac{R^2}{\rho^2}) \ , \qquad c = 1/2 \ ,$$
$$g_2(R) = R\left(1 + \frac{R^2}{\rho^2}\right)^{-\frac{E\rho}{2c\hbar}} F(a + 1 - c, b + 1 - c, 2 - c; -\frac{R^2}{\rho^2}) \ ,$$
$$h_1(R) = \left(1 + \frac{R^2}{\rho^2}\right)^{-\frac{E\rho}{2c\hbar}+1/2} F(\alpha, \beta, \gamma; -\frac{R^2}{\rho^2}) \ , \qquad \gamma = 1/2 \ ,$$
$$h_2(R) = R\left(1 + \frac{R^2}{\rho^2}\right)^{-\frac{E\rho}{2c\hbar}+1/2} F(\alpha + 1 - \gamma, \beta + 1 - \gamma, 2 - \gamma; -\frac{R^2}{\rho^2}) \ ,$$

Parameters of hypergeometric functions are given by

$$a = \frac{1}{2}\left(-\frac{E\rho}{c\hbar} + (\frac{mc\rho}{\hbar} - \frac{1}{2})\right) , \qquad b = \frac{1}{2}\left(-\frac{E\rho}{c\hbar} - (\frac{mc\rho}{\hbar} - \frac{1}{2})\right) ,$$

$$\alpha = \frac{1}{2}\left(-\frac{E\rho}{c\hbar} + 1 + (\frac{mc\rho}{\hbar} - \frac{1}{2})\right) , \qquad \beta = \frac{1}{2}\left(-\frac{E\rho}{c\hbar} + 1 - (\frac{mc\rho}{\hbar} - \frac{1}{2})\right) .$$

Let us examine the limiting procedure at $\rho \to \infty$ in $F(a, b, c; -R^2/\rho^2)$. Because

$$\frac{1}{1!}\ \frac{ab}{c}\ (-\frac{R^2}{\rho^2}) \ \ \to \ \ \frac{1}{2!}(m^2c^2/\hbar^2 - E^2/\hbar^2c^2)R^2 = -\frac{1}{2!}\ (pR)^2 \ ,$$
$$\frac{1}{2!}\ \frac{a(a+1)b(b+1)}{c(c+1)}\ (-\frac{R^2}{\rho^2}) \ \ \to \ \ +\frac{(pR)^4}{4!} \ ,$$
$$\frac{1}{3!}\ \frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2)}\ (-\frac{R^2}{\rho^2}) \ \ \to \ \ -\frac{(pR)^6}{6!} \ ,$$

and so on, we obtain the following limiting relation

$$\lim_{\rho\infty} F(a, b, c; -\frac{R^2}{\rho^2}) = \cos pr \qquad \Longrightarrow \qquad \lim_{\rho\infty} g_1(R) = \cos pr \ .$$

Similarly, we get

$$\lim_{\rho\infty} h_1(R) = \cos pr \; . \tag{98}$$

In the same manner, we arrive at two limiting relationships

$$\lim_{\rho\infty} \; pR \; g_2(R) = \sin pR \; , \qquad \lim_{\rho\infty} \; pR \; h_2(R) = \sin pR \; . \tag{99}$$

To rationalize how the finite sums (polynomials of n-order) may approximate the functions $\cos pR$ and $\sin pR$ (infinite series), we should take into account the quantization condition

$$\alpha = -n \qquad \Longrightarrow \qquad E = Mc^2 + (2n - \frac{1}{2})\frac{c\hbar}{\rho}$$

At any fixed E, as ρ increases the number n also must increase. This means, that the finite sums of n terms when ρ increases will approximate infinite series.

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